Week 10: Coactions

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Contents

1 Instructions

Please complete all the exercises. There are three 'questions' scattered throughout the sheet. Do not submit answers to these, but please bring oral solutions with you to this week's problem class.

2 Coactions

Group actions arise frequently in mathematics. The dual concept is that of a coaction, which although less intuitive is equally as useful.

Definition 1 *A (right)* ***coaction*** *of a co-H-space (A, c) on a space C is a map $\nu : C \to C \vee A$ satisfying the following two conditions.*

1. *The diagram*

$$\begin{array}{ccc} C & \xrightarrow{\nu} & C \vee A \\ & \searrow^{=} & \downarrow{\scriptstyle q_1} \\ & & C \end{array} \tag{2.1}$$

   commutes up to homotopy, where q_1 is the map pinching to the first summand.

2. *The diagram*

$$\begin{array}{ccc} C & \xrightarrow{\nu} & C \vee A \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle id_C \vee c} \\ C \vee A & \xrightarrow{\nu \vee id_A} & C \vee A \vee A \end{array} \tag{2.2}$$

commutes up to homotopy.

□

In this sheet we will see that associated with any given map $f : X \to Y$ there is a natural coaction of ΣX on the mapping cone C_f. To set notation let

$$X \xrightarrow{f} Y \xrightarrow{q} C_f \xrightarrow{\delta} \Sigma X \xrightarrow{\Sigma f} \dots \tag{2.3}$$

be the homotopy cofibration sequence associated with f. Now let

$$\nu = \nu_f : C_f \to C_f \vee \Sigma X \tag{2.4}$$

be the map induced on homotopy pushouts by the diagram

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle q}\downarrow & \overset{\psi_f}{\Rightarrow} & \downarrow & \overset{\psi_X}{\Rightarrow} & \downarrow \\ C_f & \longleftarrow & * & \longrightarrow & \Sigma X. \end{array} \tag{2.5}$$

where ψ_f, ψ_X are the canonical homotopies.

Exercise 2.1 Write down a representative for the homotopy class (2.4) and verify that it is a coaction of ΣX on C_f. □

Exercise 2.2 Show that each of the following three diagrams commutes up to homotopy.

1.

$$\begin{array}{ccc} Y & \xrightarrow{q} & C_f \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle in_1} \\ C_f & \xrightarrow{\nu} & C_f \vee \Sigma X \end{array} \tag{2.6}$$

2.

$$\begin{array}{ccc} C_f & \xrightarrow{\nu} & C_f \vee \Sigma X \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta \vee 1} \\ \Sigma X & \xrightarrow{c} & \Sigma X \vee \Sigma X \end{array} \tag{2.7}$$

3.

$$\begin{array}{ccc} C_f & \xrightarrow{\nu} & C_f \vee \Sigma X \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle q_2} \\ \Sigma X & = & \Sigma X \end{array} \tag{2.8}$$

where $c = c_X$ is the suspension comultiplication. □

The coaction ν_f is natural with respect to maps of cofiber sequences. A diagram

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle\beta}\downarrow & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle\alpha} & & \downarrow \\ Y' & \xleftarrow{f'} & X' & \longrightarrow & *. \end{array} \tag{2.9}$$

induces a map $\theta = \theta_F : C_f \to C_{f'}$ which makes

$$\begin{array}{ccc} C_f & \xrightarrow{\nu} & C_f \vee \Sigma X \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta\vee\Sigma\alpha} \\ C_{f'} & \xrightarrow{\nu'} & C_{f'} \vee \Sigma X' \end{array} \tag{2.10}$$

homotopy commute. We can see this using the equations $\theta_F\psi_f = q_{f'}F$ and $\Sigma\alpha\psi_X = \psi_{X'}\alpha$, which imply that up to homotopy both the following diagrams induce the same composite map

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle q_f}\downarrow & \overset{\psi_f}{\Rightarrow} & \downarrow & \overset{\psi_X}{\Rightarrow} & \downarrow \\ C_f & \longleftarrow & * & \longrightarrow & \Sigma X \\ {\scriptstyle\theta_F}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma\alpha} \\ C_{f'} & \longleftarrow & * & \longrightarrow & \Sigma X' \end{array} \qquad \begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle\beta}\downarrow & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle\alpha} & & \downarrow \\ Y' & \xleftarrow{f'} & X' & \longrightarrow & * \\ {\scriptstyle q_{f'}}\downarrow & \overset{\psi_{f'}}{\Rightarrow} & \downarrow & \overset{\psi_{X'}}{\Rightarrow} & \downarrow \\ C_f & \longleftarrow & * & \longrightarrow & \Sigma X'. \end{array} \tag{2.11}$$

Now, given maps $g : C_f \to Z$ and $\alpha : \Sigma X \to Z$ we define $g\dot{+}\alpha : C_f \to Z$ as the composite

$$g\dot{+}\alpha : C_f \xrightarrow{\nu} C_f \vee \Sigma X \xrightarrow{g\vee\alpha} Z \vee Z \xrightarrow{\nabla} Z. \tag{2.12}$$

Homotopies $g \simeq g'$ and $\alpha \simeq \alpha'$ induce a homotopy $g\dot{+}\alpha \simeq g'\dot{+}\alpha'$, so the construction descends to homotopy classes to give a well-defined operation

$$[C_f, Z] \times [\Sigma A, Z] \to [C_f, Z], \qquad ([g], [\alpha]) \mapsto [g]\dot{+}[\alpha] := [g\dot{+}\alpha]. \tag{2.13}$$

Exercise 2.3 Show that if $[g] \in [C_f, Z]$, then $[g]\dot{+}[*] = [g]$, and that if $[\alpha], [\beta] \in [\Sigma X, Z]$, then

$$[g]\dot{+}([\alpha] + [\beta]) = ([g]\dot{+}[\alpha])\dot{+}[\beta]. \tag{2.14}$$

□

In the sequel we will study the operation and its interaction with the exact sequence

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{q^*} [C_f, Z] \xleftarrow{\delta^*} [\Sigma X, Z] \xleftarrow{\Sigma f^*} [\Sigma Y, Z] \leftarrow \ldots \tag{2.15}$$

Exercise 2.4 For $g : C_f \to Z$ and $\alpha : \Sigma X \to Z$ as above, show that $q^*[g\dot{+}\alpha] = q^*[g]$. If also $\beta : \Sigma X \to Z$ is given, then show that $\delta^*([\alpha] + [\beta]) = (\delta^*[\alpha])\dot{+}[\beta]$. □

3 Extensions

Let $f : X \to Y$ be a map. Suppose that $g : Y \to Z$ is a map with gf null homotopic. Recall that a choice of null homotopy $F : gf \simeq *$ gives rise to an extension

$$\underline{g}_F : C_f \to Z \tag{3.1}$$

as the map defined on homotopy pushouts by

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle gf} & \overset{F}{\Rightarrow} & \downarrow \\ Z & = & Z & = & Z. \end{array} \tag{3.2}$$

We stated before that the homotopy class of $\underline{g}_F$ depends on the particular choice of the homotopy F. How does varying the homotopy change the extension? Suppose given a second null homotopy $G : gf \simeq *$. We measure the difference between F and G by means of a map

$$\delta(F, G) : \Sigma X \to Z \tag{3.3}$$

which is defined by the diagram

$$\begin{array}{ccccc} * & \xleftarrow{f} & X & \longrightarrow & * \\ \downarrow & \overset{-F}{\Rightarrow} & \downarrow{\scriptstyle gf} & \overset{G}{\Rightarrow} & \downarrow \\ Z & = & Z & = & Z. \end{array} \tag{3.4}$$

The homotopy class of $\delta(F, G)$ is called a difference element, or seperation element.

Exercise 3.1 Show that

$$\delta(F, G) = -\delta(G, F) \tag{3.5}$$

and that

$$\delta(F, F) \simeq *. \tag{3.6}$$

□

Exercise 3.2 Show that

$$\underline{g}_G \simeq \underline{g}_F \dot{+} \delta(F, G). \tag{3.7}$$

□

Exercise 3.3 Fix a space Z and consider the exact sequence of homotopy sets

$$\ldots \leftarrow [Y, Z] \xleftarrow{q^*} [C_f, Z] \xleftarrow{\delta^*} [\Sigma X, Z] \leftarrow \ldots \tag{3.8}$$

Let $k, l \in [C_f, Z]$ and show that $q^* k = q^* l$ if and only if there exists $\alpha \in [\Sigma X, Z]$ such that $l = k \dot{+} \alpha$. □

4 The General Case

The general case follows without too much additional work. If we are given a homotopy cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{4.1}$$

then we can find a homotopy $F : gf \simeq *$ inducing a homotopy equivalence $\underline{g}_F : C_f \xrightarrow{\simeq} Z$. Using this to identify Z with C_f we get a map

$$\nu = \nu_F : Z \to Z \vee \Sigma X \tag{4.2}$$

as that induced on homotopy pushouts by the diagram

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle g}\downarrow & \overset{F}{\Rightarrow} & \downarrow & \overset{\psi_X}{\Rightarrow} & \downarrow \\ Z & \longleftarrow & * & \longrightarrow & \Sigma X. \end{array} \tag{4.3}$$

The reader can check that ν is a coaction with this definition and that each of the three diagrams in Exercise 2.2 homotopy commutes when C_f is replaced by Z. We leave the task of writing down an explicit representative for ν and verifying these claims to the reader.

As above, if $u : Z \to M$ and $\alpha : \Sigma X \to M$ are maps we let $u \dot{+} \alpha$ be the composition

$$u \dot{+} \alpha : Z \xrightarrow{\nu} X \vee \Sigma X \xrightarrow{u \vee \alpha} M \vee M \xrightarrow{\nabla} M. \tag{4.4}$$

The following is then a direct consequence of Exercise 3.3.

Proposition 4.1 *Consider the homotopy cofiber sequence* (4.1). *If M is any space, then in the exact sequence*

$$[X, M] \xleftarrow{f^*} [Y, M] \xleftarrow{g^*} [Z, M] \xleftarrow{\delta^*} [\Sigma X, M] \xleftarrow{\Sigma f^*} \dots \tag{4.5}$$

it holds for $u, v \in [Z, M]$, that $g^* u = g^* v$ if and only there exists $\alpha \in [\Sigma X, M]$ such that $v = u \dot{+} \alpha$. Moreover, if $\alpha, \beta \in [\Sigma X, M]$, then $\delta^*(\alpha + \beta) = (\delta^* \alpha) \dot{+} \beta$. ∎

Thus we see that the sense of exactness of (4.5) at $[Z, M]$ is greatly improved, although it is still not as strong as the algebraic exactness enjoyed by a sequence of groups.

5 Free Homotopy Classes and the Action of $\pi_1 Y$ on $[X, Y]$

Recall that in Exercise Sheet 5 we constructed for a well-pointed space X, a strict cofiber sequence of the form

$$S^0 \hookrightarrow X_+ \to X. \tag{5.1}$$

We used the fact that the first map has a right inverse to show that the connecting map $X \to \Sigma S^0 \cong S^1$ is null homotopic, and concluded that for any space Y, the following sequence is exact in Set_*

$$0 \leftarrow \pi_0 Y \leftarrow [X, Y]_0 \leftarrow [X, Y] \leftarrow 0. \tag{5.2}$$

Of course this is not a particularly strong statement, since the exactness does not imply that the right-hand map is injective. However we know now of more structure in this sequence. Although the connecting map is null-homotopic, the sequence (5.2) retains the action of $[S^1, Y] = \pi_1 Y$ on $[X, Y]$. Applying Proposition 4.1 we conclude the following.

Proposition 5.1 *Assume that X is well-pointed and that Y is path-connected. Then every unbased map $X \to Y$ is homotopic to a based map. If $f, g : X \to Y$ are based maps which are homotopic as free maps, then there is $\alpha \in \pi_1 Y$ such that $g \simeq f \dot{+} \alpha$.* $\blacksquare$

Corollary 5.2 *If X is well-pointed and Y is simply connected, then there is a one-to-one correspondence between based and unbased homotopy classes of maps $X \to Y$. In particular, two based maps $f, g : X \to Y$ are pointed homotopic if and only if they are freely homotopic.* $\blacksquare$

How do we understance the $\pi_1 Y$-action on $[X, Y]$? Write $j : S^0 \hookrightarrow X_+$. Then the mapping cone

$$C_j = X \cup I \tag{5.3}$$

is the result of 'growing a whisker' over the basepoint. Note that the basepoint of $X \cup I$ is the far end of the whisker. The coaction

$$\nu : X \cup I \to (X \cup I) \vee S^1 \tag{5.4}$$

is the map which pinches the top half of the whisker into a circle. The canonical map $X_+ \to X$ induces a map $r : X \cup I \to X$ which collapses the whisker to a point. The map r is a free homotopy equivalence, but since the obvious map in the other direction does not respect basepoints, it is not true in general that r is a pointed homotopy equivalence. Nevertheless, if X is well-pointed, then r is a pointed homotopy equivalence (cf. *Cofiber Homotopy Equivalences*) and we get a coaction to which Proposition 4.5 applies as the composite

$$X \xrightarrow{\simeq} X \cup I \xrightarrow{\nu} (X \cup I) \vee S^1 \xrightarrow{r \vee 1} X \vee S^1 \tag{5.5}$$

where the first map is a (pointed) homotopy inverse of r.

Question *How does this compare to Hatcher's treatement in §4.A pg. 421 of Algebraic Topology? Can you use the HEP to reconcile the two approaches?* $\square$

Exercise 5.1 Assume that X is well-pointed and that (Y, m) is an H-space. Show that the action of $\pi_1 Y$ on $[X, Y]$ is trivial. $\square$

Exercise 5.2 Assume that $\varphi : X \to X'$ is a map between well-pointed spaces X, X'. Show that if Y is any space, then the induced map $\varphi^* : [X', Y] \to [X, Y]$ is $\pi_1 Y$ equivariant. If $\theta : Y \to Y'$ is any map, compute how the induced map $\theta_* : [X, Y] \to [X, Y']$ interacts with the $\pi_1 Y$ and $\pi_1 Y'$ actions. $\square$

6 Maps Between Projective Spaces

Here is another application for our improved knowledge about the structure of cofiber sequences. You will need information coming from Monday's lecture to complete the exercise in this section.

Exercise 6.1 Let $m \leq n$. Show that $[\mathbb{C}P^m, \mathbb{C}P^n] \cong \mathbb{Z}$, and that these homotopy classes are classified by their action on cohomology. $\square$

Question: *Exercise* (6.1) *shows that there is an integer* $d(f) \in \mathbb{Z}$ *associated to any self-map* $f : \mathbb{C}P^n \to \mathbb{C}P^n$. *Why is "degree" not satisfactory terminology for this integer?* $\square$

We invite the reader also to show that

$$[\mathbb{R}P^m, \mathbb{R}P^n] \cong \mathbb{Z}_2 \tag{6.1}$$

for $m < n$. However the reader will probably find that the answer may be different for $m = n$.

Question: *What do you need to know to compute* $[\mathbb{H}P^m, \mathbb{H}P^n]$? $\square$